Field :→ Every physical quantity can be expressed as a continuous function of the position of a point in the region of space, such a function is called point function and the region in which it specifies the physical quantity is known as a field.

Ex. – Electric field, magnetic field etc.

a) Scalar field :→ It is represented by continuous scalar function, the value of which is characterised by a single value of a scalar quantity at every point in space.

Ex. – Electric potential field is a scalar field.

b) Vector field :→ A vector field is represented at every point in space by a continuous vector function which is specified by a single vector quantity of definite magnitude and direction at any point in space.

Ex. – Electric field, magnetic field and vector field $A = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+3z)\hat{k}$

Del operator or differential operator :→ $\vec{\nabla}$

It is a vector and defined as

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

here $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the three mutually perpendicular axis.

It has no physical significance by itself, but acquires significance when operate on a vector and scalar field.

1. ## Gradient of a Scalar Field :→

Let $\phi(r)$ be a scalar field. Then the gradient of scalar field $\phi(r)$ is represented by grad $\phi(r)$ and mathematically defined as

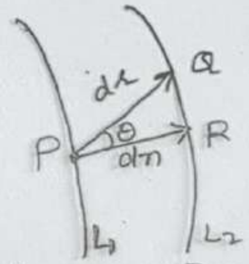
$$\text{grad}\,\phi(r) = \vec{\nabla}\phi(r), \quad \text{here } \vec{\nabla} \text{ is del operator.}$$

$$\Rightarrow \text{grad}\,\phi(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(r)$$

$$\boxed{\text{grad}\,\phi(r) = \hat{i}\frac{\partial \phi(r)}{\partial x} + \hat{j}\frac{\partial \phi(r)}{\partial y} + \hat{k}\frac{\partial \phi(r)}{\partial z}}$$

Hence gradient of a scalar fn. is a vector quantity.

Magnitude and Direction of $\nabla\phi$. Let us consider two level surfaces L_1 and L_2 through two close points P and Q distance dr apart (Fig. 1.), with the values of the scalar function ϕ and $(\phi + d\phi)$ respectivley. Let $PR = dn$ be the normal to the surface L_1 at P.



As L_2 is a level surface, the value of the scalar function is the same at R as at Q.

The rate of change of ϕ along the normal PR is equal to $\dfrac{\partial \phi}{\partial n}$.

Now $$dn = dr\cos\theta = \hat{n}\,.\,dr,$$

where $\hat{n}$ is the unit vector normal to the surface L_1 at P so that

$$d\phi = \frac{\partial \phi}{\partial n}\,dn = \frac{\partial \phi}{\partial n}\,\hat{n}\,.\,dr \qquad \text{---}(1)$$

As we know,

$$\nabla\phi\,.\,dr = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right).\,(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= d\phi$$

or $$d\phi = \nabla\phi\,.\,dr \qquad \text{---}(2)$$

Combining equations (1) and (2), we may write

$$\therefore \nabla\phi\,.\,dr = \frac{\partial \phi}{\partial n}\,\hat{n}\,.\,dr \qquad \text{---}(3)$$

As dr is arbitrary, Equ. (3) gives

$$\nabla\phi = \frac{\partial \phi}{\partial n}.\hat{n} \Rightarrow \boxed{\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial n}\hat{n}}$$

Hence, the magnitude of $\nabla\phi$ is equal to the maximum rate of change of scalar ϕ and is directed along this maximum rate of change.

1c

Gradient of Electric Potential :→

The electric potential is given as $u(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}$$

Now take gradient of above expression, we have

$$\vec{\nabla} U = \vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\varepsilon_0} q \vec{\nabla}\left(\frac{1}{r}\right)$$

Since $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ we have

$$\vec{\nabla}\left(\frac{1}{r}\right) = \vec{\nabla}(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(\hat{i}\,2x + \hat{j}\,2y + \hat{k}\,2z)$$

$$= -r^{-3}\vec{r} = -\frac{1}{r^2}\frac{\vec{r}}{r}$$

$$= -\frac{\hat{r}}{r^2} \qquad (\because \vec{r} = |\vec{r}|\hat{r})$$

$$\therefore \quad \vec{\nabla} U = \frac{1}{4\pi\varepsilon_0} q\left(\frac{-\hat{r}}{r^2}\right) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\vec{\nabla} U = -\vec{E} \qquad \left(\because \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)$$

2. Divergence of a vector field :→ Let $\vec{E}$ is a vector field the divergence of $\vec{E}$ is represented as Div $\vec{E}$. Divergence of a vector field is a scalar quantity.

The divergence of vector field $\vec{E}$ (div $\vec{E}$) is defined as the limiting value of the ratio of the closed surface integral to the volume enclosed by the surface over which integration is carried out, when the volume tends to zero, i.e.,

$$\boxed{\vec{\nabla}\cdot\vec{E} = div\,\vec{E} = \underset{v\to 0}{\mathrm{Lt}}\frac{1}{v}\oiint \vec{E}\cdot\vec{ds}} \qquad \text{——— (1)}$$

where v is the volume enclosed by the source S over which integration is carried out. In vector form div $\vec{E}$ is represented by $\vec{\nabla}\cdot\vec{E}$.

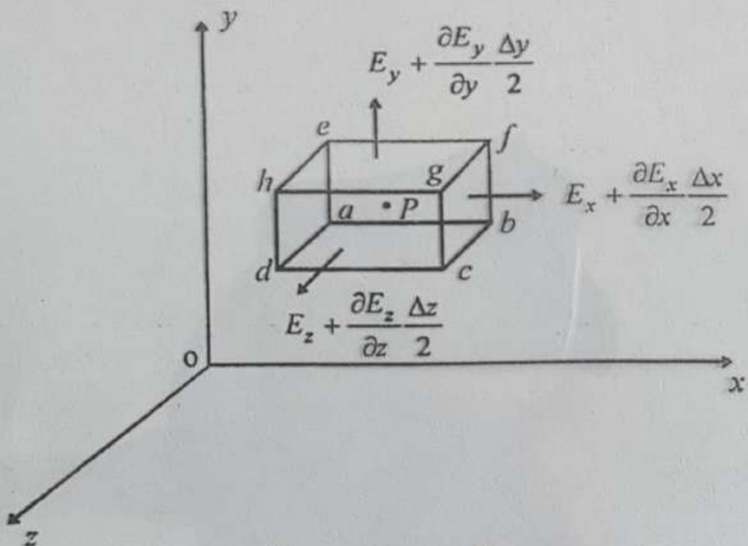
If a vector function $\vec{E}$ spreads out, i.e., diverges from a point, then it has a positive divergence at that point and point acts as a source of the field $\vec{E}$. Indeed $\vec{\nabla}\cdot\vec{E}$ can be taken to be a measure of the spreading out of the field. On the other hand, if the field converges to a point then $\vec{\nabla}\cdot\vec{E}$ will be negative at that point because the point acts as a sink for the field $\vec{E}$. Finally, if the vector field $\vec{E}$ neither converges nor diverges then $\vec{\nabla}\cdot\vec{E} = 0$.

Divergence in Terms of Cartesian Co-ordinates

Suppose a parallelopiped *abcefgh* is placed in an electric field $\vec{E}$ as shown in Fig [redacted]. We have considered the sides of the parallelopiped parallel to the coordinate axes system. Let Δx, Δy and Δz respectively be the sides of the parallelopiped along x, y and z axes. Therefore, the volume of parallelopiped is $\Delta x \Delta y \Delta z$.

Let $P(x, y, z)$ be at the mid point of the parallelopiped and the strength of the vector field at P be (x, y, z). Let $\frac{\partial E_x}{\partial x}, \frac{\partial E_y}{\partial y}, \frac{\partial E_z}{\partial z}$ be the rates of change of the components along the x, y and z axes.

Since the surface area of *cbfg* is very small, we have assumed constant value of the x-component of $\vec{E}$ and its value is $E_x + \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}$ as the distance between point P and surface *cbfg* is $\frac{\Delta x}{2}$. Similarly, the strength of x-components on the surface *adhe* $= E_x - \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}$



Also
$$\iint_{cbfg} \vec{E}\cdot\vec{ds} = \iint_{cbfg} \left[E_x + \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}\right] ds$$
$$= \left[E_x + \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}\right] \Delta y\, \Delta z \qquad \ldots(2)$$

and
$$\iint_{adhe} \vec{E}\cdot\vec{ds} = \iint_{adhe} \left[E_x - \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}\right] ds$$
$$= \left[-E_x + \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}\right] \Delta y\, \Delta z \qquad \ldots(3)$$

Negative sign is appearing with $\iint \vec{E}\cdot\vec{ds}$ because the directions of x-component *adhe* of $\vec{E}$ and the area element $\vec{ds}$ are antiparallel.

In a same way, we can calculate the value of $\iint \vec{E}\cdot\vec{ds}$ for other surfaces then the sum of all the components will give the value of the surface integral over the closed surface of the parallelopiped; i.e

$$\oiint \vec{E}\cdot\vec{ds} = \left[\iint_{cbfg}\vec{E}\cdot\vec{ds} + \iint_{bfea}\vec{E}\cdot\vec{ds} + \iint_{fchg}\vec{E}\cdot\vec{ds} + \iint_{ghdc}\vec{E}\cdot\vec{ds} + \iint_{aehd}\cdot\vec{E}\cdot\vec{ds} + \iint_{cbad}\vec{E}\cdot\vec{ds}\right]$$

$$= \left[E_x + \frac{\partial E_x}{\partial x}\frac{\Delta x}{2}\right]\Delta y\,\Delta z - \left[E_z - \frac{\partial E_z}{\partial z}\frac{\Delta z}{2}\right]\Delta x\,\Delta y$$

$$+ \left[E_x + \frac{\partial E_y}{\partial y}\frac{\Delta y}{2}\right]\Delta x\,\Delta z + \left[E_y + \frac{\partial E_z}{\partial z}\frac{\Delta z}{2}\right]\Delta x\,\Delta y$$

$$- \left[E_y - \frac{\partial E_x}{\partial y}\frac{\Delta x}{2}\right]\Delta y\,\Delta z + \left[E_y + \frac{\partial E_y}{\partial y}\frac{\Delta y}{2}\right]\Delta x\,\Delta z \qquad \ldots(4)$$

$$= \frac{\partial E_x}{\partial x}\Delta x\,\Delta y\,\Delta z + \frac{\partial E_y}{\partial y}\Delta x\,\Delta y\,\Delta z + \frac{\partial E_z}{\partial z}\Delta x\,\Delta y\,\Delta z$$

$$= \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{dz}\right]\Delta x\,\Delta y\,\Delta z \qquad \ldots(5)$$

But $\quad div\,\vec{E} = \underset{v\to 0}{\mathrm{Lt}}\frac{1}{v}\oiint \vec{E}\cdot\vec{d}s$

$$= \frac{1}{\Delta x\,\Delta y\,\Delta z}\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right]\Delta x\,\Delta y\,\Delta z$$

$$\Rightarrow \qquad div\,\vec{E} = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] \qquad \ldots(6)$$

which is the expression for $div\,\vec{E}$ in terms of Cartesian co-ordinates.

So $div\,\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}E_x + \hat{j}E_y + \hat{k}E_z\right)$$

$$\boxed{div\,\vec{E} = \vec{\nabla}\cdot\vec{E}}$$

Curl of a Vector field :→

Let $\vec{B}$ is any vector field, then curl of $\vec{B}$ is denoted by Curl $\vec{B}$ and is defined as the ratio of maximum value of line integral of that vector field along a closed path and the surface area s enclosed by the closed curve at that point such that area s tends to be zero.

$$|\text{Curl } \vec{B}| = \lim_{S \to 0} \frac{\oint \vec{B} \cdot \vec{dl}}{S}$$

The direction of curl $\vec{B}$ is same as that of area vector $\vec{S}$.

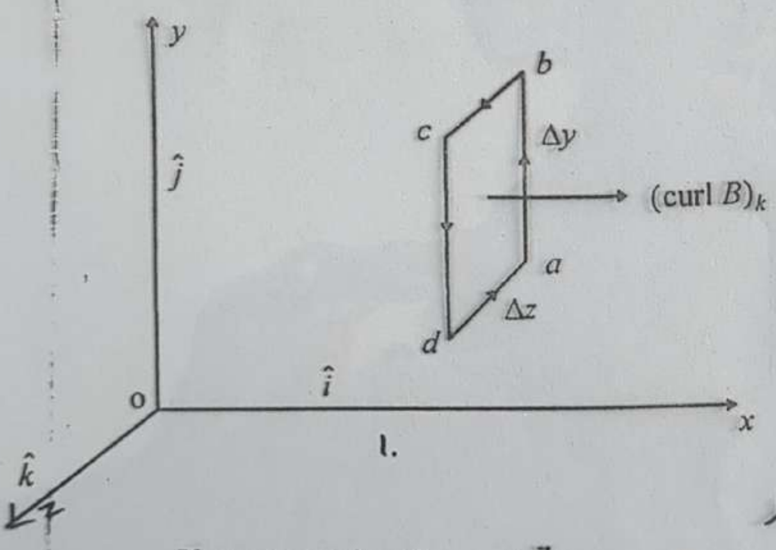
Hence curl of a vector field is also a vector quantity.

Curl in Terms of Cartesian Co-ordinate :→

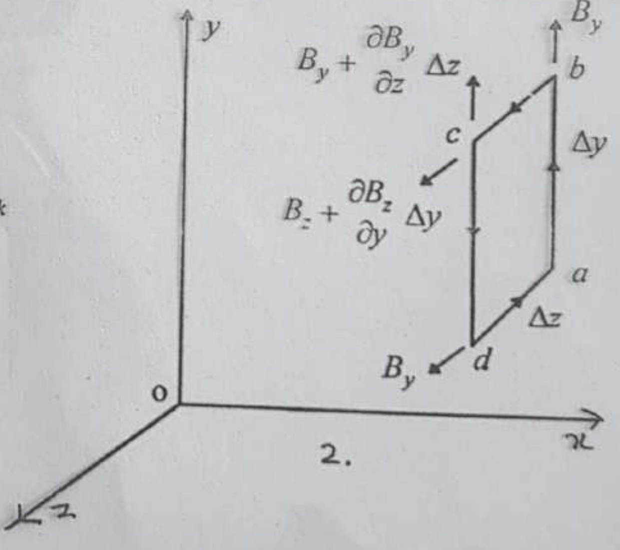
consider a rectangular loop abcd along x-axis, having area S_x. Let dimension of loop along y and z axes by Δy and Δz. Suppose the co-ordinates of the corners a, b, c and d are $a(x, y, z)$, $b(x, y+\Delta y, z)$, $c(x, y+\Delta y, z+\Delta z)$ and $d(x, y, z+\Delta z)$.

Now consider a vector field $\vec{B}(\vec{r})$, the the component of curl $\vec{B}$ in the x-direction is

$$(\text{Curl } B)_x = (\text{Curl } \vec{B}) \cdot i = \lim_{S \to 0} \frac{1}{S_x} \oint \vec{B} \cdot \vec{dl} \qquad (1)$$



1.



2.

If we consider the curl $\vec{B}$ across the loop *abcd*, then the above equation can be written as

$$(\text{curl } \vec{B})_x = \underset{S_x \to 0}{\text{lt}} \frac{1}{S_x} \left[\int_{ab} \vec{B} \cdot d\vec{l} + \int_{bc} \vec{B} \cdot d\vec{l} + \int_{cd} \vec{B} \cdot d\vec{l} + \int_{da} \vec{B} \cdot d\vec{l} \right] \qquad (2)$$

Let the value of y component of $\vec{B}$ be B_y. Then on the side ab

$$\int_{ab} \vec{B} \cdot d\vec{l} = \int_{ab} B_y dl = By\Delta y$$

On the side cd, the value of y component is $B_y + \frac{\partial B_y}{\partial z}\Delta z$, where $\frac{\partial B_y}{\partial z}$ is the rate of change of B_y along z-axis.

$$\text{Hence} \quad \int_{cd} \vec{B}\cdot d\vec{l} = -\int\left(B_y + \frac{\partial B_y}{\partial z}\Delta z\right)dl = -\left(B_y + \frac{\partial B_y}{\partial z}\Delta z\right)\Delta y$$

Negative sign appears because $d\vec{l}$ is directed in the direction opposite to the Y-component.

$$\text{Similarly} \int_{bc}\vec{B}\cdot d\vec{l} = \left(B_z + \frac{\partial B_z}{\partial y}\Delta y\right)\Delta z \quad \text{and} \quad \int_{da}\vec{B}\cdot d\vec{l} = -B_z\Delta z$$

$\therefore$ From eqn. (2) and putting $S_x = \Delta z \Delta y$

$$\text{Curl}\,\vec{B} = \frac{1}{\Delta y\,\Delta z}\left[B_y\Delta y - \left(B_y + \frac{\partial B_y}{\partial z}\Delta z\right)\Delta y + \left(B_z + \frac{\partial B_z}{\partial y}\Delta y\right)\Delta z - B_z\Delta z\right]$$

$$= \frac{1}{\Delta y \Delta z}\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right]\Delta y \Delta z$$

$$\therefore \quad (\text{curl}\vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \qquad \text{——— (3)}$$

Similarly,

$$(\text{curl}\vec{B})_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \qquad \text{——— (4)}$$

$$\text{and} \quad (\text{curl}\,\vec{B})_y = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}. \qquad \text{——— (5)}$$

$$\text{Thus.} \quad \text{curl}\,\vec{B} = \hat{i}(\text{curl}\vec{B})_x + \hat{j}(\text{curl}\vec{B})_y + \hat{k}(\text{curl}\vec{B})_z$$

$$= \hat{i}\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right] + \hat{j}\left[\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right] + \hat{k}\left[\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right]$$

$$\text{curl}\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{——— (6)}$$

In term of del operator $\vec{\nabla}$, curl of vector field $\vec{B}$ is written as

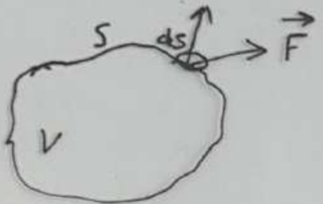
$$\text{Curl}\,\vec{B} = \vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{\nabla}\times\vec{B} = \text{curl}\,\vec{B} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z\right)$$

Gauss's Divergence Theorem ⇒

It state that surface integral of any vector field $\vec{F}$ over any closed surface S is equal to the volume integral of the divergence of that vector field over the volume enclosed by the closed surface.

let $\vec{F}$ is vector field
and $\vec{ds}$ small surface element.



Then

$$\boxed{\begin{aligned} \oiint_S \vec{F}\cdot\vec{ds} &= \int_V \operatorname{div}\vec{F}\,dv \\ &= \int_V \vec{\nabla}\cdot\vec{F}\,dv \end{aligned}}$$

Stoke's Theorem ⇒

It state that line integration of tangential component of any vector field $\vec{F}$ around a closed curve C is equal to surface integral of the normal component of curl of $\vec{F}$ over the surface bounded by the closed curve C.

$$\Rightarrow \oint_C \vec{F}\cdot\vec{dl} = \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{ds} = \iint_S \text{Curl}\,\vec{F}\cdot\vec{ds}$$

Types of Vector Fields ⇒

1. Lamellar Vector field ⇒
if it can be expressed as the gradient of of a scalar field. ⇒ $\vec{E} = -\vec{\nabla}u$
so $\vec{E}$ is lamellar field.

The line integral of lamellar field is independent of path. Also line integration over closed curve of a lamellar field is zero, which are called conservative fields.
So lamellar fields are conservative fields.

2. Solenoidal vector Field ⇒
A vector field $\vec{B}$ is said to be solenoidal if its divergence is zero.

$$\boxed{\Rightarrow \vec{\nabla}\cdot\vec{B} = \text{div}.\vec{B} = 0}$$

3. Irrotational vector fields :→

If $\vec{\nabla}\times\vec{E} = 0$, then $\vec{E}$ is called irrotational vector field.

4. Rotational vector field :→

If $\vec{\nabla}\times\vec{E} \neq 0$, then $\vec{E}$ is called rotational vector field.

Usefull Results :→

i div grad $u = \vec{\nabla}\cdot\vec{\nabla}u = \nabla^2 u$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

ii Curl grad $u = \vec{\nabla}\times\vec{\nabla}u = 0$

because it is cross product of two similar vectors.

iii div curl $\vec{B} = 0$

$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$

iv Curl curl $\vec{B} = \vec{\nabla}\times\vec{\nabla}\times\vec{B}$

$= \text{grad div}\vec{B} - \nabla^2\vec{B}$

$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$

Numericals :→

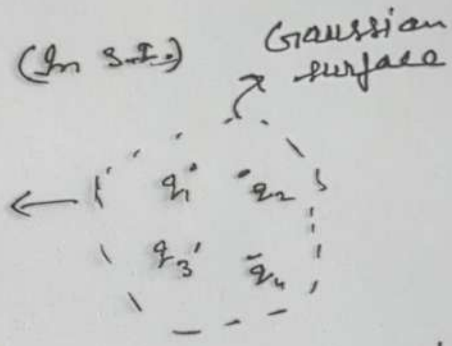
1. If $\phi = 3x^2y - y^2z^2$, find the value of grad ϕ at point $(1,-2,-1)$
2. obtain the gradient of i r^m ii $\frac{1}{r}$ iii $\log r$
3. Show that curl of grad ϕ is zero $\Rightarrow \vec{\nabla}\times\vec{\nabla}\phi = 0$
4. Find the value of c if vector $A = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+cz)\hat{k}$ is solenoidal.
5. If $A = xz^3\hat{i} + 2yz^4\hat{k} - 2x^2yz\hat{j}$, then find $\vec{\nabla}\cdot\vec{A}$ & $\vec{\nabla}\times\vec{A}$
6. If w is a constant vector and $\vec{v} = \vec{w}\times\vec{r}$, then show that div $v = \vec{\nabla}\cdot\vec{v} = 0$
7. Find the constt a, b, and c so that the vector $A = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

1.1. Gauss's Law & Application

1. Gauss's Law ÷ It state that electric flux ϕ_E through any closed surface is equal to $1/\varepsilon_0$ times the net charge q enclosed by the surface.

$$\phi_E = \oint_S \vec{E}.d\vec{S} = \frac{q}{\varepsilon_0} \text{ (In S.I.)}$$

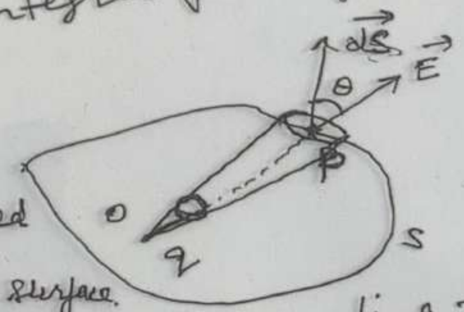
$$\phi_E = \oint_S \vec{E}.d\vec{S} = \frac{q_1+q_2+q_3+q_4}{\varepsilon_0}$$



$$\Rightarrow \boxed{\oint_S \vec{E}.d\vec{S} = \frac{q}{\varepsilon_0}}$$

It is integral form of Gauss law

Proof ÷



Point charge $+q$ at origin O inside a closed surface called Gaussian surface. Let dS is small patch of area surrounding the point P. Let $OP = r$. $\vec{dS}$ ~~ds~~ is the area vector. Let $\vec{E}$ is electric field at point P due to charge q.

The electric flux outward through the small area dS is

$$d\phi_E = \vec{E}.\vec{dS} = E\,dS\cos\theta \qquad \text{——— 1.}$$

where θ is the angle between $\vec{E}$ & $\vec{dS}$

$$\text{Now } \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{——— 2.}$$

$$\therefore d\Phi_E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dS\cos\theta \qquad \text{——— 3.}$$

The solid angle subtended by small area dS at point O is given as

$$d\Omega = \frac{dS\cos\theta}{r^2}$$

6. (1k)

$$\Rightarrow d\phi_E = \frac{1}{4\pi\varepsilon_0} q\, d\Omega \quad \text{—— 4.}$$

The total electric ~~field~~ flux ϕ_E through the surface S is

$$\Phi_E = \frac{q}{4\pi\varepsilon_0}\int d\Omega \quad \text{—— 5.}$$

The solid angle subtended by the entire closed surface S at O is

$$\oint d\Omega = 4\pi \quad \text{—— 6}$$

$$\Rightarrow \Phi_E = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{4\pi\varepsilon_0} 4\pi$$

$$\Phi_E = \oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0} \quad \text{—— 7}$$

$$\Rightarrow \oint \varepsilon_0 \vec{E}\cdot\vec{dS} = q$$

$$\boxed{\text{or} \int \vec{D}\cdot\vec{dS} = q} \quad \text{—— 8.}$$

Here $\vec{D} = \varepsilon_0 \vec{E}$ is electric displacement vector for free space

Equations 7 & 8 represent the Integral form of Gauss Law.

* If there are several charges enclosed by Gaussian surface like $q_1, q_2, -q_3 \ldots$ then total flux through the surface is

$$\Phi_E = \oint_S \vec{E}\cdot\vec{dS} = \frac{1}{\varepsilon_0}(q_1 + q_2 - q_3 \ldots) = \frac{1}{\varepsilon_0}\sum_i q_i$$

* If there is no charge enclosed by the surface then $\boxed{\Phi_E = 0}$

2. Differential form of Gauss Law :-

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0} \quad \text{——— (1)}$$

From Gauss divergence theorem,

$$\oint_S \vec{E}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{E}\, dv \quad \text{——— (2)}$$

$$\Rightarrow \iiint \vec{\nabla}\cdot\vec{E}\, dv = \frac{q}{\varepsilon_0} \quad \text{——— (3)}$$

let q is the total charge contained in the volume. If ρ is the charge density, then total charge q is written in term of ρ as

$$q = \iiint_V \rho\, dv \quad \text{——— (4)}$$

Putting in 3. we have

$$\iiint \vec{\nabla}\cdot\vec{E}\, dv = \frac{1}{\varepsilon_0}\iiint_V \rho\, dv = \iiint_V \frac{\rho}{\varepsilon_0}\, dv$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \boxed{\vec{\nabla}\cdot\vec{D} = \rho} \quad \text{——— (5)}$$

$$\Rightarrow \vec{\nabla}\cdot\varepsilon_0\vec{E} = \rho$$

This is differential form of Gauss Law.

3. Laplace's and Poisson's equation :-

The differential form of Gauss law is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Also $\vec{E} = -\vec{\nabla}U$

$$\Rightarrow \vec{\nabla}\cdot(-\vec{\nabla}U) = \frac{\rho}{\varepsilon_0}$$

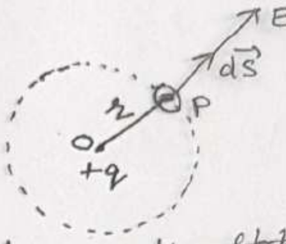
$$\boxed{\nabla^2 U = -\frac{\rho}{\varepsilon_0}} \longrightarrow \text{Poisson's Equation}$$

For charge free space $\rho = 0$

$$\Rightarrow \boxed{\nabla^2 U = 0} \longrightarrow \text{Laplace's Equation}$$

Applications of Gauss's Law :→

1. Coulomb's Law from Gauss's Law :→



let +q charge at O.
Draw Gaussian surface of radius r in the form of a sphere. The point P at which the $\vec{E}$ is to be determined lies on the ~~sph~~ surface of Gaussian Surface.
The electric flux through the Gaussian surface is

$$\phi_E = \oint_S \vec{E}\cdot\vec{ds} = \oint_S E\,ds\cos 0 = \oint_S E\,ds$$

$$\phi_E = E\oint_S ds = E\,4\pi r^2$$

But according to Gauss's law, $\phi_E = \frac{q}{\varepsilon_0}$

$$\Rightarrow \quad E\,4\pi r^2 = \frac{q}{\varepsilon_0} \quad \Rightarrow \boxed{E = \frac{q}{4\pi\varepsilon_0 r^2}}$$

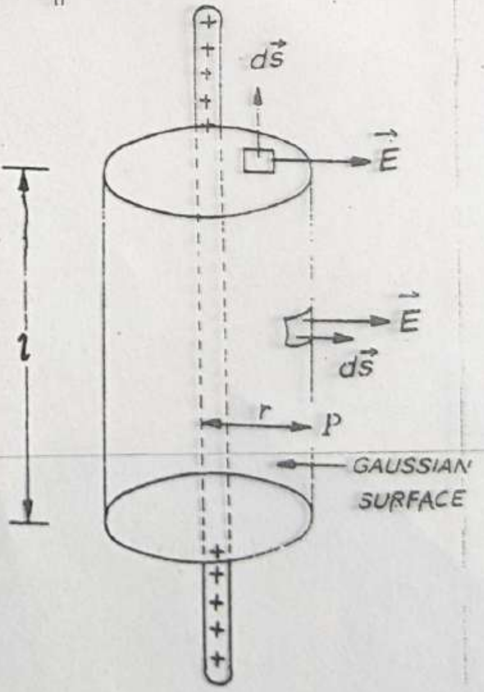
It is the electric field due to charge +q at any point P.
It is Coulomb's law.
If another charge q_0 is placed at P, then force experienced by charge q_0 is $F = \frac{1}{4\pi\varepsilon_0}\frac{q\,q_0}{r^2}$ (Coulom's Law)

2. Electric Field due to an infinite line of charge :→

let q is total charge.

$\lambda = \frac{q}{l}$ → linear charge density

Consider Gaussian surface in cylindrical form of length l and radius r.



The electric flux due to curved surface

$$= \oint \vec{E}\cdot d\vec{S} = E\oint d\vec{S} = E\,.\,(2\pi rl)$$

The electric flux due to each plane face

$$= \oint \vec{E}\cdot d\vec{S} = \oint E\,dS\cos 90° = 0$$

∴ Total flux through the Gaussian surface

$$\phi = E(2\pi rl) = \frac{\text{charge}}{\epsilon_0}$$

$$E(2\pi rl) = \frac{\lambda l}{\epsilon_0} \qquad \left(\because \lambda = \frac{q}{l}\right)$$

$$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}}$$

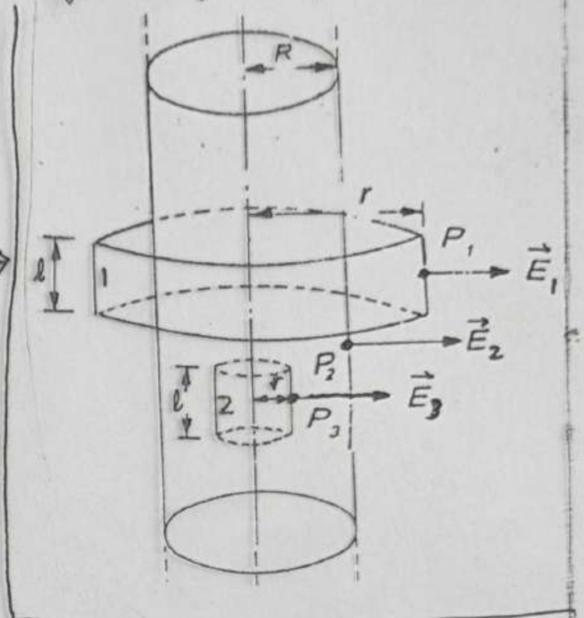
1n

3. Electric Field due to an infinite long charged cylinder :→

$\lambda = \frac{q}{l}$ (linear charge density)

$\Rightarrow q = \lambda l$

R – radius of cylinder.



i At an external point P_1 $(r > R)$:→

Let $\vec{E}_1$ is electric field at P_1

$\vec{E}_1$ & $\vec{ds}$ are ∥ at curved surface.

Electric flux through the Gaussian surface 1 is

$$\phi = \int \vec{E}_1 \cdot \vec{ds} = \int E_1 ds \cos 0^\circ + \int E_1 ds \cos 90^\circ = \int E_1 ds = E_1 (2\pi r l)$$

Also $\phi = \frac{q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$

$$\Rightarrow E_1 (2\pi r l) = \frac{\lambda l}{\varepsilon_0} \Rightarrow \boxed{E_1 = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

ii On the surface of charged cylinder P_2 $(r = R)$:→

Let P_2 be the point on the surface of charge distribution, where electric field intensity is $\vec{E}_2$ and the electric flux is :

$$\phi = E_2 (2\pi r l) = \frac{\lambda l}{\epsilon_0}$$

$$\boxed{E_2 = \frac{\lambda}{2\pi \epsilon_0 R}}$$

(iii) At an internal point ($r < R$). Let P_3 be the point at a distance $r < R$ from the axis of the cylinder, the electric field intensity is $\vec{E}_3$ considers coaxial cylindrical surface of length l and radius r through P_3.

The electric flux through the Gaussian surface is :

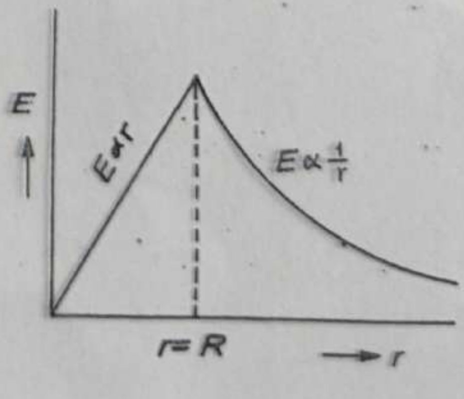
$$\phi = E_3 (2\pi r l)$$

The charge contained within the Gaussian surface is $(\pi r^2 l)\,\rho$ where ρ is surface charge density

$$\therefore \quad \phi = E_3 (2\pi r l) = \frac{(\pi r^2 l)\rho}{\epsilon_0}$$

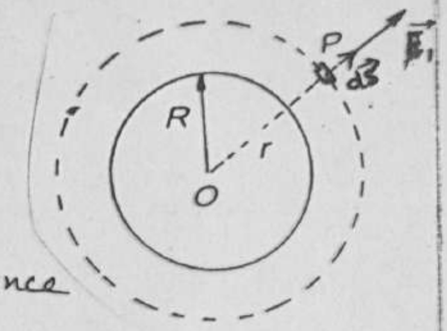
$$\boxed{E_3 = \frac{r\rho}{2\epsilon_0}} \quad \text{... (68)}$$

The variation of electric field with distance r is shown in Fig



4. Electric field due to uniformly charged sphere ⇒

Consider a sphere of Radius R and O as centre



i At external point $P(r > R)$ ⇒

Let ρ is volume charge density

$$\Rightarrow \boxed{\rho = \frac{q}{\frac{4}{3}\pi R^3} \text{ or } q = \frac{4}{3}\pi R^3 \rho}$$

Let an external point P at a distance r from centre of charge sphere.

Now draw a concentric Gaussian spherical surface through P.

The electric flux through the Gaussian surface is

$$\phi = \oint \vec{E}_1 \cdot \vec{ds} = \oint E_1 ds \cos 0° = E_1 \oint ds = E_1 4\pi r^2$$

But $\phi = \frac{q}{\varepsilon_0} \Rightarrow E_1 4\pi r^2 = \frac{q}{\varepsilon_0}$

$$\boxed{E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3 \rho}{r^2}}$$

ii At the surface ⟶ Let E_2 is electric field at surface,

Here $r = R$

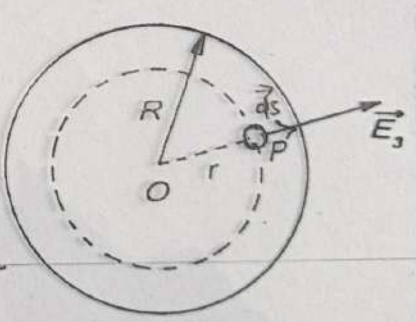
$$\Rightarrow \boxed{E_2 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2}}$$

Hence $\vec{E}_1$ & $\vec{E}_2$ are same if total charge is assumed to be concentrated at centre.

iii At internal point ⇒ Let $\vec{E}_3$ is electric field inside sphere

Point P lies inside the sphere.

Draw Gaussian surface through P.



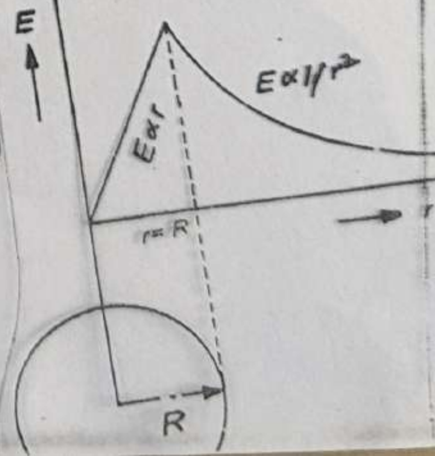
a) For conducting sphere ⇒

$$\phi = \oint E_3 \cdot ds = E_3(4\pi r^2)$$

~~Here~~ Let q' is charge enclosed by Gaussian surface. In case of conductor the whole charge resides on surface of spherical conductor.

$$\Rightarrow q' = 0 \Rightarrow E_3(4\pi r^2) = 0$$

$$\Rightarrow \boxed{E_3 = 0}$$



b) For non-conducting sphere ⇒ ($r < R$)

here $q' = \frac{4}{3}\pi r^3 \rho$

$$\therefore \phi = E_3 \cdot 4\pi r^2 = \frac{q'}{\varepsilon_0}$$

$$\Rightarrow E_3 = \frac{q'}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi r^3 \rho}{r^2}$$

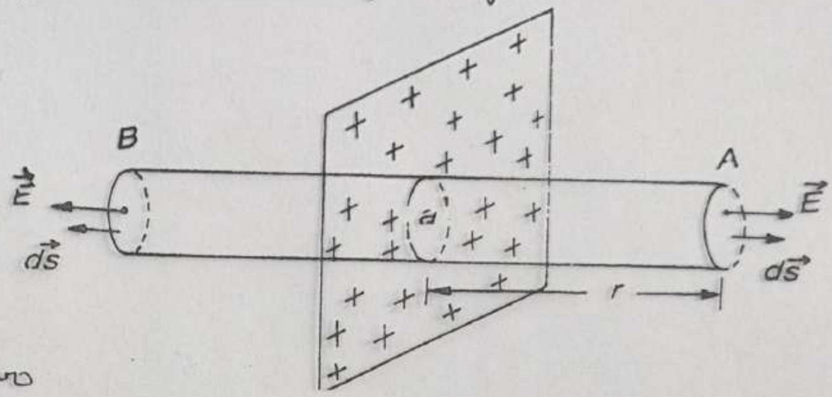
5. Electric Field due to an infinite plane sheet of charge ⇒

let σ is surface charge density.

Consider a Gaussian surface in cylindrical shape through the charged sheet with its face // to the sheet.

let 'a' is face area of cylinder.

The flux through the curved surface of cylinder is zero.



The flux through the two plane ends is

$$\phi_1 = \int \vec{E}\cdot\vec{ds} + \int \vec{E}\cdot\vec{ds}$$

$$= E\int ds + \int E\,ds = Ea + Ea = 2Ea$$

The flux through the curved surface $\phi_2 = 0$

Total flux through the Gaussian surface

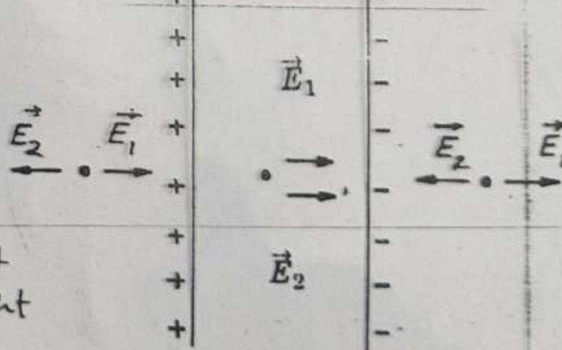
$\phi = 2Ea$ and $\phi = \frac{q}{\varepsilon_0} = \frac{\sigma a}{\varepsilon_0}$ (From Gauss's law)

$$\Rightarrow 2Ea = \frac{\sigma a}{\varepsilon_0} \Rightarrow \boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

⇒ $\vec{E}$ is independent of r and ⊥ to plane of sheet.

6 Field due to two parallel sheets of charge ⇒

a) having equal and opposite charges ⇒



σ is charge density.

$\vec{E}_1 = \frac{\sigma}{2\varepsilon_0}$, $\vec{E}_2 = \frac{\sigma}{2\varepsilon_0}$

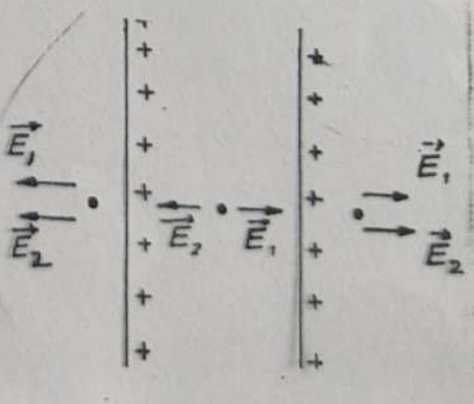
$\vec{E} = \vec{E}_1 + \vec{E}_2 =$

$$\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0}$$ At left and right point

At the point between the sheet,

$$\vec{E} = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

b) having equal and same charges ⇒



At left and right side of sheets

$$\vec{E} = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

At point between the sheets -

$$\boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0}$$

Potential due to monopole or a point charge :→

Consider an isolated point charge q.

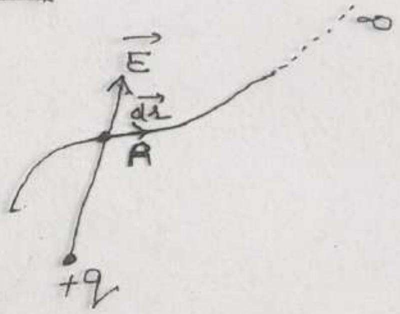
The electric potential at point in electric field is defined as the work done by an external agent in moving a unit +ve charge (test charge) from infinity to that point against the electric field and force.

let W — work done
q_0 — +ve charge
V → potential at any point A

then $\boxed{V = \frac{W}{q_0}}$ $\boxed{1V = \frac{1J}{1C}}$

Let q_0 — +ve test charge.
$\vec{E}$ → electric field.
q → point (monopole) charge which produces

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \longrightarrow (1)$$



Force experienced by $q_0 = q_0 \times \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q q_0}{r^2} = \vec{F}$ —— (2)

let q_0 moves a small distance, then done against the electric force is

$$dW = -\vec{F}\cdot\vec{dr} = -q_0\,\vec{E}\cdot\vec{dr} \qquad \text{—— (3)}$$

Total work done in moving charge q_0 from ∞ to poin A is

$$W_{\infty A} = \int_{\infty}^{A} dW = -\int_{\infty}^{A} q_0\,\vec{E}\cdot\vec{dr} = -\int_{-\infty}^{A} q_0 E\,dr\cos\theta \text{ —— (4)}$$

here θ is angle between $\vec{E}$ and $\vec{dr}$.

If $\vec{E}$ and $\vec{dr}$ are in same direction. then

$$W_{\infty A} = -\int_{\infty}^{A} q_0 E\,dr = -\int_{+\infty}^{A} \frac{q_0 q}{4\pi\varepsilon_0 r^2}\,dr$$

$$W_{\infty A} = -\frac{q_0 q}{4\pi\varepsilon_0}\int_{\infty}^{A}\frac{dr}{r^2} = -\frac{q_0 q}{4\pi\varepsilon_0}\left|-\frac{1}{r}\right|_{\infty}^{A} = -\frac{q_0 q}{4\pi\varepsilon_0}\left(-\frac{1}{r_A} - \left(-\frac{1}{r_\infty}\right)\right)$$

$$W_{\infty A} = -\frac{q_0 q}{4\pi\varepsilon_0}\left(\frac{1}{r_\infty} - \frac{1}{r_A}\right) \qquad \left\{\because \frac{1}{r_\infty} = \frac{1}{\infty} = 0\right\}$$

$$= -\frac{q_0 q}{4\pi\varepsilon_0}\left(-\frac{1}{r_A}\right) = \frac{q_0 q}{4\pi\varepsilon_0 r_A}$$

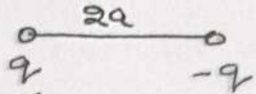
So electric potential at point A due to monopole is

$$\boxed{V = \frac{W_{\infty A}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_A}}$$

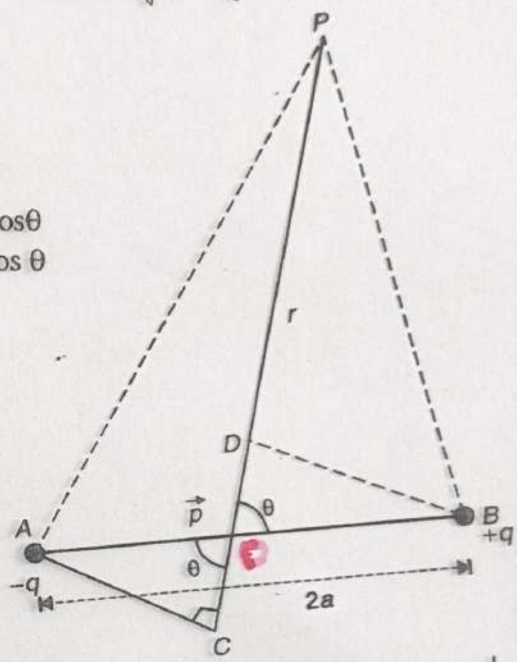
Potential due to an electric dipole ⇛

A dipole consists of two equal and opposite charges separated by a small distance.

The dipole moment $\boxed{\vec{p} = q \times 2a}$ and point from -ve charge to +ve charge.



Let AB be an electric dipole of length $2a$ and let P be any point where $OP = r$ and let θ be the angle between r and the dipole axis. Polar co-ordinate of $P = (r, \theta)$



In ΔOAC, $\cos\theta = \frac{OC}{OA} = \frac{OC}{a}$

$\therefore$ $OC = a\cos\theta$

similarly, $OD = a\cos\theta$

if $r >> a$,

then $PA = PC = PO + OC = r + a\cos\theta$

and $PB = PD = PO - OD = r - a\cos\theta$

If V be the potential at P due to electric dipole then

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{PB} - \frac{q}{PA}\right)$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{(r - a\cos\theta)} - \frac{1}{(r + a\cos\theta)}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{r + a\cos\theta - r + a\cos\theta}{r^2 - a^2\cos^2\theta}\right]$$

or $$V = \frac{q}{4\pi\varepsilon_0}\,\frac{2a\cos\theta}{r^2 - a^2\cos^2\theta}$$

$$V = \frac{1}{4\pi\varepsilon_0}\,\frac{2q.a\cos\theta}{r^2 - a^2\cos^2\theta}$$, if $\vec{p} = 2qa$ = dipole moment

Then $$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{p\cos\theta}{r^2\left(1 - \frac{a^2}{r^2}\cos\theta\right)}$$

As $r >> a$

$\therefore$ $\frac{a}{r} << 1$, so neglect $\frac{a^2}{r^2}\cos^2\theta$ as compared to 1,

we have, $$\boxed{V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{p\cos\theta}{r^2}}$$ ⇛ $\boxed{V \propto \frac{1}{r^2} \text{ (dipole)}}$

Special cases

(i) When the point P lies on the axial line of the dipole on the side of positive charge, $\theta = 0$, $\therefore$ $\cos\theta = 1$

So from eq. (1)

$$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{p}{r^2} \quad ...(2)$$

(ii) When the point P lies on the axial line of the dipole but on the side of negative charge, $\theta = 180°$ or $\cos\theta = -1$

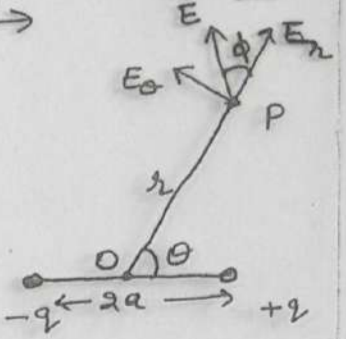
$\therefore$ $$V = \frac{-1}{4\pi\varepsilon_0}\cdot\frac{p}{r^2} \quad ...(3)$$

(iii) When the point P lies on the equitorial line of the dipole, $\theta = 90°$ or $\cos\theta = 0$

$\therefore$ $$V = 0 \quad ...(4)$$

which means that no work is done in bringing a charge from infinity to the dipole along the perpendicular bisector of the dipole.

Electric Field from potential due to a dipole $\Rightarrow$



$$\vec{E} = -\vec{\nabla} V \quad \text{——} (1)$$

and $$V = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2} \quad \text{——} (2)$$

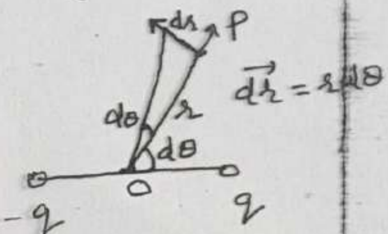
If point P moves along OP, r changes but θ remains constant.
If point P moves right angle to OP, then θ changes but r remains constant.

As $V = V(r, \theta)$, therefore $\vec{E}$ at point P must be calculated in the direction of r increasing as well as in the direction of θ increasing.

Let E_r is field intensity in the direction of r increasing, then

$$E_r = -\frac{dV}{dr} = -\frac{d}{dr}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right)$$

$$\boxed{E_r = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}} \quad \text{——} (3)$$



Now a small distance perpendicular to r is $dr = r d\theta$. ——(4)

If E_θ is the electric field intensity at point P in the direction of θ increasing, then

$$\vec{E}_\theta = -\frac{dV}{dr} = -\frac{dV}{r d\theta}$$

$$= -\frac{1}{r}\left[\frac{d}{d\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right)\right]$$

$$\vec{E}_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3} \quad \text{——} (4)$$

The resultant electric field $\vec{E} = \sqrt{E_r^2 + E_\theta^2}$

$$E = \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{2p\cos\theta}{r^3}\right)^2 + \left(\frac{p\sin\theta}{r^3}\right)^2\right]^{1/2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left(\sin^2\theta + 4\cos^2\theta\right)^{1/2} = \frac{p}{4\pi\varepsilon_0 r^3}\left(1+3\cos^2\theta\right)^{1/2}$$

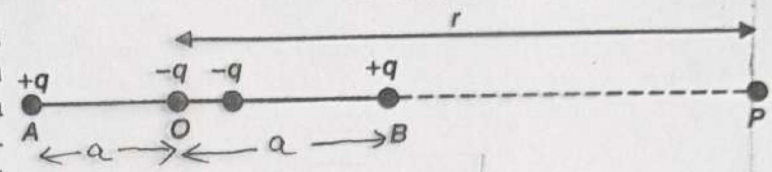
$$\boxed{\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(1+3\cos^2\theta\right)^{1/2}}$$

and $$\tan\phi = \frac{E_\theta}{E_r} = \frac{1}{2}\tan\theta$$

Electric Field due to quadrupole :→

An electric quadrupole consists of *two electric dipoles placed end to end along the same line.*

OA and OB are two dipoles each of length a. It may be noted that though the total charge on the system as a whole is zero, *the potential and intensity are not zero.* Let us consider a point P along the axis of the quadrupole.



It is clear from the Fig. the potential at the point P is given by,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(r-a)} - \frac{q}{r} - \frac{q}{r} + \frac{q}{r+a}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{r(r+a) - 2(r^2-a^2) + r(r-a)}{r(r^2-a^2)}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{r^2 + ra - 2r^2 + 2a^2 + r^2 - ra}{r(r^2-a^2)}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{2a^2}{r(r^2-a^2)}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{2qa^2}{r(r^2-a^2)}$$

Now $Q = 2qa^2$, the quadrupole moment of charge distribution

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r(r^2-a^2)} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^3(1 - a^2/r^2)}$$

If $r >> a$, $\quad \frac{q}{r} << 1$

$\therefore \quad \frac{a^2}{r^2}$ can be neglected as compared to 1

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^3}$$

So in case of

(*i*) Monopole (single charge)

$$V \propto \frac{1}{r}$$

(*ii*) Dipole

$$V \propto \frac{1}{r^2}$$

(*iii*) Quadrupole

$$V \propto \frac{1}{r^3}$$

Multipole Expansion :→

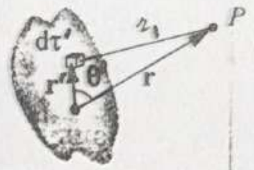
We want to develop a systematic expansion for the potential of an arbitrary charge distribution in the power of $\frac{1}{r}$.

Let $\rho(r')$ is volume charge ~~distributi~~ density over volume τ. The potential at poin P at distance r from origin O is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1} = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r_1}\rho(r')\,d\tau' \qquad (1)$$

Using the law of cosines,

$$r_1^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right],$$



or

$$r_1 = r\sqrt{1+\epsilon_1}$$

where

$$\epsilon_1 \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right). \qquad (2)$$

For points well outside the charge distribution, ϵ is much less than 1, and this invites a binomial expansion:

$$\frac{1}{r_1} = \frac{1}{r}(1+\epsilon_1)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon_1 + \frac{3}{8}\epsilon_1^2 - \frac{5}{16}\epsilon_1^3 + \dots\right), \qquad (3)$$

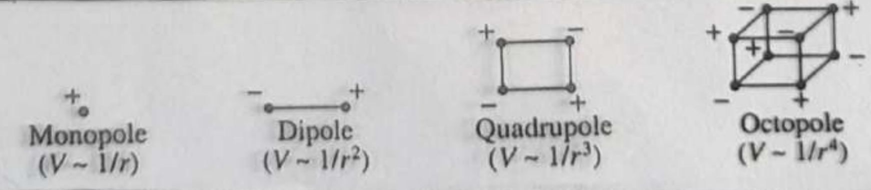
or, in terms of r, r', and θ':

$$\begin{aligned}\frac{1}{r_1} &= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2\right.\\ &\qquad \left. - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \dots\right]\\ &= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\theta') + \left(\frac{r'}{r}\right)^2 (3\cos^2\theta' - 1)/2\right.\\ &\qquad \left. + \left(\frac{r'}{r}\right)^3 (5\cos^3\theta' - 3\cos\theta')/2 + \dots\right]. \end{aligned} \qquad (4)$$

$$\begin{aligned}\therefore\ V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau'\right.\\ &\qquad \left. + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \dots\right]. \end{aligned} \qquad (5)$$

This is the desired result—the **multipole expansion** of V in powers of $1/r$.

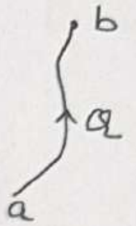
So



Work and Energy in Electrostatics :→

1. Let us consider a charge Q is planced in electric field $\vec{E}$. Then force experienced by charge Q is

$$\vec{F} = Q\vec{E}$$



Let us apply a force opposite to electric force $= -Q\vec{E} = \vec{F_1}$

Small work done on charge Q is

$$\boxed{dw = \vec{F_1}\cdot\vec{dl} = -Q\vec{E}\cdot\vec{dl}} \quad \text{———} (1)$$

This work done is stored in the form of kinetic energy.

Now, the line integral of electric field gives the electric potential.

$$\Rightarrow \boxed{V = \int \vec{E}\cdot\vec{dl}} \quad \text{———} (2)$$

$$\therefore \; W = \int dw = \int_a^b -Q\vec{E}\cdot\vec{dl} = -Q\int_a^b \vec{E}\cdot\vec{dl}$$

$$W = -Q\left[V(a) - V(b)\right] = Q\left[V(b) - V(a)\right]$$

$$\boxed{W = Q\left[V(b) - V(a)\right]}$$

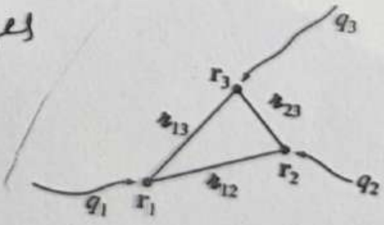
The work done = charge × potential difference.

2. **Energy of point charge distribution :→**

It is the amount of work required to assemble an entire collection of point charges of a system.

consider collection of three point charges (q_1, q_2 & q_3) r_1, r_2 and r_3 are position vectors of point charges.

Electrostatic energy is work required to place the all charges at their position in the system.



i. No work is required to place the first charge q_1 at r_1.

$$\Rightarrow W_1 = 0 \quad \text{———} (1)$$

ii. To bring the charge q_2 at r_2, the required work is $W_2 = q_2 V_1 = q_2\left(\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{12}}\right)$ ——— (2)

iii Now bring the charge q_3 at position r_3,
then $W_3 = q_3 V_1 + q_3 V_2$

$$W_3 = q_3 (V_1 + V_2) = q_3 \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_{23}} \right)$$

$$W_3 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{r_{23}} \quad \text{---} (3)$$

In this way, the total ~~charge~~ work necessary to assemble the first three charges is

$$W = W_1 + W_2 + W_3 = 0 + \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}} + \frac{q_1 q_3}{4\pi\varepsilon_0 r_{13}} + \frac{q_2 q_3}{4\pi\varepsilon_0 r_{23}}$$

$$\boxed{W = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]} \quad \text{---} (4)$$

Equ. (4) gives the electrostatic energy of a system having three charges.

If there are n charges in system, then energy of system is written as

$$W = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \cdots\cdots + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \cdots \right]$$

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{r_{ij}}$$

Here $j > i$ for not to count the same pair twice.

or $$W = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}} \quad \text{---} (5)$$

In equ. (5), we have counted the each pair twice and multiplied the expression by $\frac{1}{2}$.

Equ. (5) may be written as

$$W = \cancel{\frac{1}{4\pi\varepsilon_0}} \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{\substack{j=1 \\ j=i}}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)$$

$$\boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)}$$

Energy is required to dismental the system. So it is called electrostatic energy of the system.

Dielectrics :→

are bassically insulators in which all the electrons are tightly bounded to the nuclei of atoms. Hence the conductivity is very low.

Ex — Glass, plastic, mica oil etc.

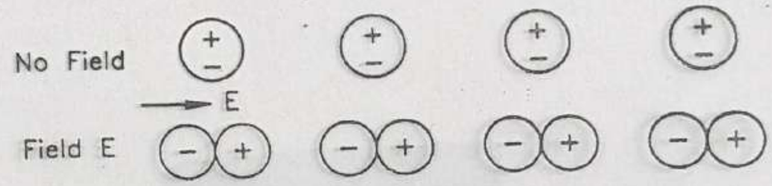
Non-Polar and Polar Dielectrics :→

Non-Polar molecules :→

A non-polar molecule is one in which the centre of gravity of the positive charges coincides with the centre of gravity of the negative charges. Symmetrical molecules (*e.g.*, H_2, N_2 and O_2) are *non-polar*. A non-polar molecule has obviously a zero electric dipole moment.

If the dielectric (with non-polar molecules) is placed in an electric field, the charge centres of a non-polar molecule become displaced. The molecules are then said to be polarized by the field and are called *induced dipoles*.

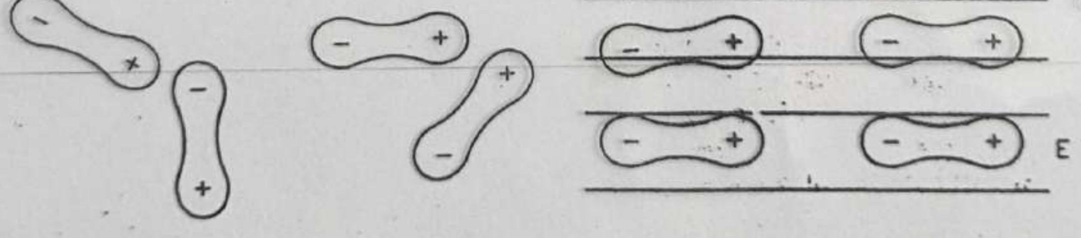
The molecules thus acquire an induced electric dipole moment in the direction of the field.



Polar Molecules :→

A polar molecule is one in which the centre of gravity of the positive charges is separated from the centre of gravity of the negative charges by a finite distance. The polar molecule is thus an electric dipole and has an intrinsic permanent electric-dipole moment. Ex. — HCl, H_2O

In the absence of any external electric field, the individual dipoles are oriented at random and no net dipole moment is observed in the dielectric. When an electric field is applied, the forces on a dipole give rise to a torque, whose effect is to orient the dipole along the direction of electric field.
Howeve the allignment is not complete due to thermal agitation of molecules

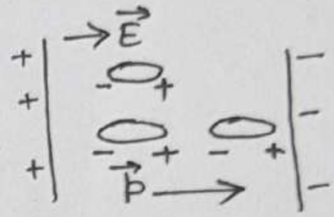


Randomly aligned (*b*) Aligned in the direction of field

Dielectric Polarisation :→

When the atoms (polar or non-polar) are placed in external electric field, they aquire an induced electric dipole moment in the direction of applied field. This process is known as dielectric polarisation.

The induced dipole moment $\vec{p}$ in a molecule is directely proportional to the applied electric field $\vec{E}$.



$$\Rightarrow \quad \vec{p} \propto \vec{E} \quad \text{or} \quad \boxed{\vec{p} = \alpha \vec{E}}$$

Here α is constt. and is called <u>atomic polarisability</u>

Electric Polarisation Vector :→ $\vec{P}$

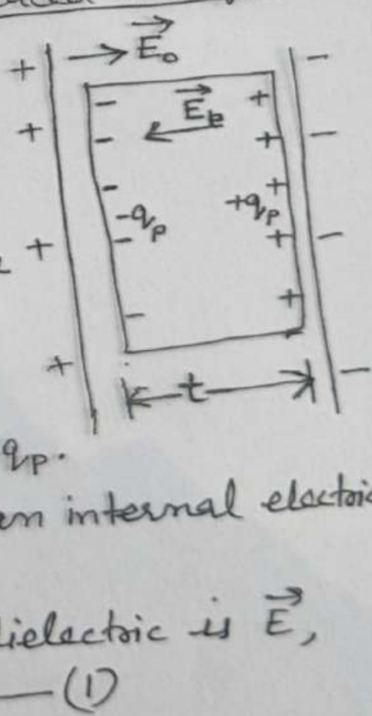
$\vec{P}$ is defined as the induced dipole moment per unit volume within a dielectrics.

Let n — no. of molecules per unit volume

$\vec{p}$ → induced dipole moment per molecule.

then $\boxed{\vec{P} = n\vec{p}} = n\alpha\vec{E}$ Unit $=$ coul/m^2

Electric Polarisation vector $\vec{P}$ and induced charges :→



Let a dielectric slab is placed in an external electric field $\vec{E_0}$.

Let A — face area of slab.

t — thickness of slab.

-ve and +ve charges of equal magnitude apear on the two faces of slab.

These charges are called $\boxed{\text{induced charges or bound charges.}}$ or q_p.

These induced charges will produce an internal electric field $\vec{E_p}$ opposite to $\vec{E_0}$.

So net electric field inside the dielectric is $\vec{E}$,

$$\boxed{\vec{E} = \vec{E_0} - \vec{E_p}} \quad \text{——— (1)}$$

Now dipole moment of slab = charge × t = $q_p \times t$ ——— (2)

Let σ_p is bound charge surface density,

then $\sigma_p = \dfrac{q_p}{A} \Rightarrow q_p = \sigma_p \times A$ ——— (3)

From equ. (2) and (3)

Dipole moment of slab $= \sigma_p \times A \times t$ —— (4)

The volume of slab is equal to $A \times t$.

$\therefore$ The dipole moment per unit volume of slab is

$$\vec{P} = \frac{\text{Dipole moment of slab}}{\text{Volume of slab}} = \frac{\sigma_p \times A \times t}{A \times t}$$

$$\boxed{\vec{P} = \vec{\sigma_p}}$$

$\therefore$ Electric polarisation vector is equal to the ~~in~~ surface bound charge density of the slab.

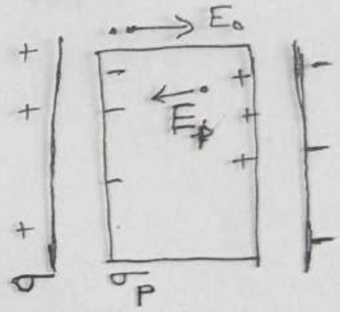
<u>Displacement vector $\vec{D}$</u> :->

$\vec{E_0} \longrightarrow$ applied field.

$\vec{E_p} \longrightarrow$ induced field due to polarisation.

$\vec{E} \rightarrow$ net electric field.

then $\vec{E} = \vec{E_0} - \vec{E_p}$ —— (1)



$\sigma \rightarrow$ surface charge density of free charges.

$\sigma_p \rightarrow$ surface charge density of bound charges.

$\vec{E_0} = \frac{\sigma}{\varepsilon_0}$ (field between two plates)

$\vec{E_i} = \frac{\sigma_p}{\varepsilon_0}$

$\Rightarrow \vec{E} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_p}{\varepsilon_0} \Rightarrow \sigma = \varepsilon_0 \vec{E} + \sigma_p$

$\{$ as $\vec{P} = \sigma_p \}$

or $\boxed{\sigma = \varepsilon_0 \vec{E} + \vec{P}}$

We calls this <u>free charge density</u> as <u>displacement vector $\vec{D}$.</u>

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

Units of $D = C/m^2$

Here $\vec{D}$ is connected with the free charges only.

$\vec{P} \longrightarrow$ with induced or polarisation or bound charges.

$\vec{E} \longrightarrow$ with free and bound charges

[Hence the electric displacement vector is concerns with free charges only.]

Relative Permittivity and susceptibility :→

Relative permittivity ε_r is defined as the ratio of permittivity of ~~space to~~ medium to permittivity of free space.

$\varepsilon \rightarrow$ permittivity of medium
$\varepsilon_0 \rightarrow$ permittivity of free space
$\varepsilon_r \rightarrow$ relative permittivity.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = K \rightarrow \text{dielectric constant.}$$

$$\Rightarrow \boxed{\varepsilon = \varepsilon_r \varepsilon_0} \quad \text{———} (1)$$

Now, the electric polarisation vector $\vec{P}$ is proportional to the electric field inside the dielectrics.

$$\Rightarrow \vec{P} \propto \vec{E} \Rightarrow \boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}} \quad \text{———} (2)$$

Here χ_e is known as electric susceptibility of the medium and is dimensionless constt. ε_0 is introduce to make χ_e dimenssionless.

we know that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$$\Rightarrow \vec{D} = \varepsilon_0 \left[\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right] = \varepsilon_0 \left(\vec{E} + \frac{\varepsilon_0 \chi_e \vec{E}}{\varepsilon_0}\right)$$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E}(1 + \chi_e)} \quad \text{———} (4)$$

But $\vec{D}$ is also proportion to the electric field $\vec{E}$

$$\Rightarrow \vec{D} \propto \vec{E} \text{ or } \boxed{\vec{D} = \varepsilon \vec{E}} \quad \text{———} (5)$$

where ε – permittivity of free space.

From equ. 4 and 5,

$$\boxed{\varepsilon = \varepsilon_0(1 + \chi_e) \text{ or } \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e}$$

or $\boxed{\varepsilon_r = 1 + \chi_e} = K$ (dielectric constt.

Also, the Gauss law in dielectric ~~of~~ permittivity ε is $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon} \Rightarrow \vec{\nabla} \cdot \varepsilon \vec{E} = \rho$

or $\boxed{\vec{\nabla} \cdot \vec{D} = \rho} \rightarrow$ Gauss law in dielectrics.

Molecular Field in Dielectrics →
or Claussius – Mossotti Equation :→

The molecular field which is responsible for polarising a molecul of the dielectric material is known as the molecular field or local field.

Concentrating on a particular molecule, molecular field is the field produced by external sources and by all polarised molecules of dielectric except the molecule of consideration.

Let $\vec{E}_{loc}$ → local electric field ~~produ~~ experience by a particular molecule.

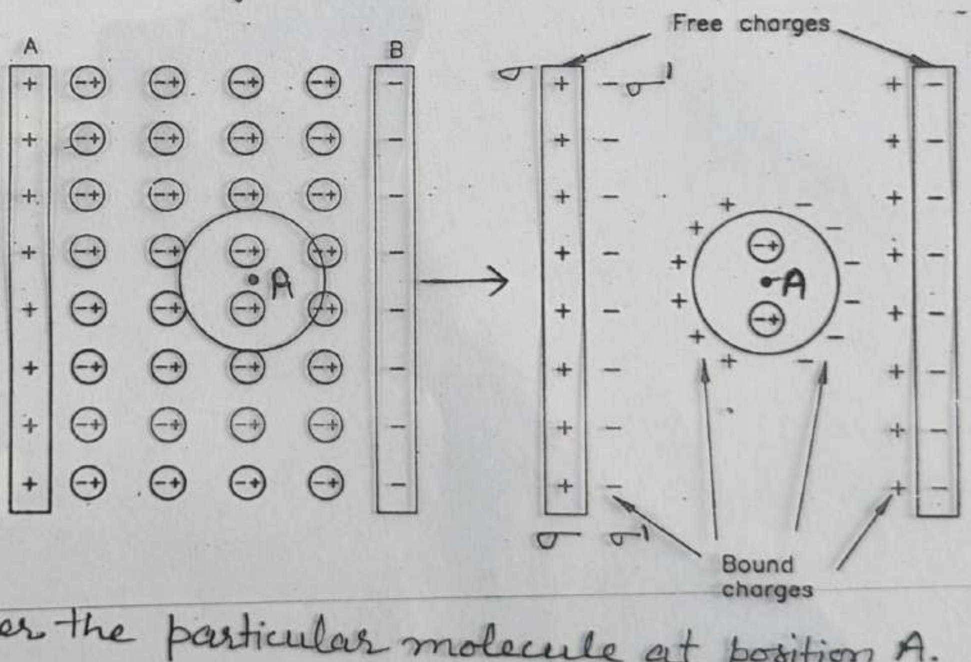
∴ dipole moment produced in a molecule is

$$\vec{p} = \alpha \vec{E}_{loc} \quad \text{--- (1)}$$

and dielectric polarisation vector $\vec{P}$ is

$$\boxed{\vec{P} = n\vec{p} = n\alpha \vec{E}_{loc}} \quad \text{--- (2)}$$

here n – is no. of molecules per unit volume.



Now consider the particular molecule at position A. This reference molecule at A surrounded by spherical cavity called Lorentz sphere.

Divide the dielectric into two parts:–

1. Dielectric outside the sphere (Lorentz) is considered as continuum of dipoles.
2. The molecules inside the sphere are treated as individual molecules.

The molecular field or the local field acting on the molecule A consists of four components.

$$\vec{E}_{loc} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \vec{E}_3 \qquad \text{——— (3)}$$

where

i $\vec{E}_0$ is external applied electric field.

$$\boxed{\vec{E}_0 = \frac{\sigma}{\varepsilon_0}} \qquad \text{——— (4)}$$

ii $\vec{E}_1$ is the electric field due to induced or bound charges

$$\Rightarrow \vec{E}_1 = -\frac{\sigma'}{\varepsilon_0} \text{ or } -\frac{\vec{P}}{\varepsilon_0} \qquad \text{——— (5)}$$

iii $\vec{E}_2$ is the electric field due to polarised charges on the surface of Lorentz sphere

and $\vec{E}_2 = \frac{\vec{P}}{3\varepsilon_0}$ ——— (6)

iv $\vec{E}_3$ is the field due to permanent dipoles lying within the Lorentz sphere.

Here we have assumed that dielectric is made of non-polar molecules

$$\Rightarrow \vec{E}_3 = 0 \qquad \text{——— (7)}$$

$$\Rightarrow \vec{E}_{loc} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma'}{\varepsilon_0} + \frac{\vec{P}}{3\varepsilon_0} = \vec{E} + \frac{\vec{P}}{3\varepsilon_0} \qquad \text{——— (8)}$$

Putting in equ. (2)

$$\vec{P} = n\alpha\left(\vec{E} + \frac{\vec{P}}{3\varepsilon_0}\right) \qquad \text{——— (9)}$$

we know that $\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P}$

or $\vec{P} = (\varepsilon - \varepsilon_0)\vec{E}$

or $\boxed{\vec{E} = \vec{P}/(\varepsilon - \varepsilon_0)}$ ——— (10)

From (9) and (10)

$$\vec{P} = n\alpha\left[\frac{\vec{P}}{\varepsilon - \varepsilon_0} + \frac{\vec{P}}{3\varepsilon_0}\right] = n\alpha\vec{P}\left[\frac{2\varepsilon_0 + \varepsilon}{3\varepsilon_0(\varepsilon - \varepsilon_0)}\right]$$

or $\frac{3\varepsilon_0}{n\alpha} = \frac{\varepsilon + 2\varepsilon_0}{\varepsilon - \varepsilon_0}$ or $\boxed{\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} = \frac{n\alpha}{3\varepsilon_0}}$

But $\varepsilon = \varepsilon_r\varepsilon_0 = K\varepsilon_0$

$$\Rightarrow \boxed{\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{n\alpha}{3\varepsilon_0}} \text{ or } \boxed{\frac{K-1}{K+2} = \frac{n\alpha}{3\varepsilon_0}}$$

* Maxwell's Modification in Ampere's Law :→

$$\vec{\nabla} \times \vec{B} = \mu_0 J \quad \rightarrow \text{Ampere's Law} \quad \text{——} \ (1)$$

Taking divergence on both side of equ. (1)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} \quad \text{——} \ (2)$$

Since the divergence of curl is always zero.

From Equ. (2) ,∴ $\vec{\nabla} \cdot \vec{J} = 0$ —— (3)

But it is true only for steady current as we know it from equation of continuity.

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{——} \ (4)$$

which is true for varying currents.

So Ampere's Law is not valid for non-steady currents. Maxwell removed this flaw by introducing the concept of displacement current

From Gauss Law,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$$

or $$\frac{\partial \rho}{\partial t} = \varepsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} \quad \text{——} \ (5)$$

Add $\vec{\nabla} \cdot \vec{j}$ on both side of equ. (5),

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{J} + \varepsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \left[\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right] \quad \text{——} \ (6)$$

From equs. (4) and (6) , we have

$$\vec{\nabla} \cdot \left[\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right] = 0 \quad \text{——} \ (7)$$

So for varying currents

$$\nabla \cdot \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0$$

Therefore, Maxwell proposed that $\vec{J}$ in Ampere's Law should by replaced by

$$\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{——— (8)}$$

The term $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ is known as displacement current.

So modified Ampere's Law is

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \qquad \text{— (9)}$$

or $$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right]$$

which is valid for varying current also.

From equ. (9), it is clear that changing electric field induces magnetic field which is opposite to the Faraday's Law.

Maxwell's Equations

These are basic equations of Electromagnetism,

Integral Form

i $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \leftrightarrow$ (Gauss's Law) $\leftrightarrow \oint_S \vec{E}\cdot\vec{ds} = \frac{1}{\varepsilon_0}\int_V \rho\, dv$

ii $\vec{\nabla}\cdot\vec{B} = 0 \leftrightarrow$ (no-name) $\leftrightarrow \oint_S \vec{B}\cdot\vec{ds} = 0$

iii $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \leftrightarrow$ (Faraday's Law) $\leftrightarrow \int \vec{E}\cdot\vec{dl} = \iint_S \frac{\partial \vec{B}}{\partial t}\cdot ds$

iv $\vec{\nabla}\times\vec{B} = \mu_0\left(J + \varepsilon_0\frac{\partial E}{\partial t}\right) \leftrightarrow \int \vec{B}\cdot\vec{dl} = \int_S \left(J + \varepsilon_0\frac{\partial E}{\partial t}\right)\cdot ds$

(Ampere's Law with Maxwell's modification)

Reference → Rakesh Dogra

* The above equation can be modified under different ~~as~~ conditions or mediums

1. For free space, there is no flow of charge and no charged particles present in the space. Hence

$$J = 0 \quad \& \quad \rho = 0$$

So Maxwell's equations are

i $\vec{\nabla}\cdot\vec{E} = 0$ ii $\vec{\nabla}\cdot\vec{B} = 0$

iii $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$ iv $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$

2. For medium, where no free charge or free current present,

$$\varepsilon_0 = \varepsilon \quad \& \quad \mu_0 = \mu$$

$$\rho = 0 \quad \& \quad J = 0$$

Electromagnetic Wave Equation :→

In free space,

$\rho = 0$ and $J = 0$

So Maxwell's Equation for free space are

i $\vec{\nabla}.\vec{E} = 0$ ii $\vec{\nabla}.\vec{B} = 0$

iii $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ iv $\vec{\nabla}\times\vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

It is a set of first order partial differential equations.

Now take the curl on both sides of equa. iii

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla\times\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) \quad \text{——— (1)}$$

But $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E}$ ——— (2)

From equ. i $\vec{\nabla}.\vec{E} = 0$.

So, from (2), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} \quad \text{——— (3)}$$

From equs. (1) and (3),

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) \quad \text{——— (4)}$$

Using equ. iv, we have

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}} \quad \text{——— (5)}$$

It is electromagnetic wave equation in term of electric field for free space.

In a similar way, taking curl on both side of equation iv, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu_0\varepsilon_0 \frac{\partial E}{\partial t}\right) \qquad \text{——— (8)}$$

But $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2 B$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} \quad \{\because \vec{\nabla}\cdot\vec{B}=0\} \qquad \text{——— (9)}$$

$\therefore$ From (8) and (9), we have

$$-\nabla^2\vec{B} = \mu_0\varepsilon_0\left(\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow -\nabla^2\vec{B} = \mu_0\varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \qquad \text{——— (10)}$$

Using equa. iii, one gets

$$-\nabla^2\vec{B} = \mu_0\varepsilon_0\left(-\frac{\partial^2\vec{B}}{\partial t^2}\right)$$

or $$\boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}} \qquad \text{——— (11)}$$

It is e.m. wave equation in term of $\vec{B}$ for free space.

Therefore, electromagnetic wave equation for $\vec{E}$ and $\vec{B}$ may be written as

$$\boxed{\begin{aligned} &\nabla^2 E - \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} = 0 \\ \text{and} \quad &\nabla^2 B - \mu_0\varepsilon_0\frac{\partial^2 B}{\partial t^2} = 0 \end{aligned}} \qquad \text{——— (12)}$$

But the general wave equa. ~~equat~~ is

$$\nabla^2 y - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{——— (13)}$$

, where c is velocity of wave

From equs. (12) and (13) it is clear that in Maxwell's wave equation, the wave propagate with velocity

$$\left| c = 1/\sqrt{\mu_0\varepsilon_0} \right| \qquad \text{——— (14)}$$

$\therefore$ e.m. wave velocity is

$$\left.\begin{aligned} c &= \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \text{for free space} \\ \text{and } c &= \frac{1}{\sqrt{\mu \cdot \varepsilon}} \quad \text{for medium} \end{aligned}\right] \quad \text{——} \quad (15)$$

Also the quantity $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ has the dimension of impedence which is called characteristics impedence.

So $\mu_0 = 4\pi \times 10^{-7}$

& $\varepsilon_0 = 8.85 \times 10^{-12}$

$$\therefore \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99 \times 10^{8} \text{ m/sec}$$

and $\sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377$ ohm